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A radiating charged particle in Einstein's universe

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Abstract. A metric is obtained which in the vicinity of the source reduces to the metric describing the field of a radiating charged particle and which in the absence of the source reduces to the metric of the well known Einstein static universe.

1. Introduction

The generalisation of DeSitter's and Friedmann's cosmological solutions with a point mass is well known in the literature (McVittie 1933, Tolman 1934). Vaidya and Shah (1957) have discussed a radiating mass particle in an expanding universe, while Patel and Shukla (1974) have discussed a radiating charged particle in an expanding universe. However, a solution describing a radiating charged particle in Einstein's universe has not yet been found. The purpose of this paper is to fill this gap.

Einstein's model of the universe is the simplest geometrical model for an isotropic and homogeneous universe which is static. The geometry of this universe is described by the metric

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 - \frac{(x\,dx + y\,dy + z\,dz)^2}{R^2 - (x^2 + y^2 + z^2)} \tag{1.1}$$

where R is a constant.

We carry out the following transformation from (x, y, z, t) to the coordinates (r, α, β, u):

$$x = R\sin(r/R)\sin\alpha\cos\beta, \qquad y = R\sin(r/R)\sin\alpha\sin\beta,$$
$$z = R\sin(r/R)\cos\alpha, \qquad u = t - r. \tag{1.2}$$

Under (1.2) Einstein's metric (1.1) transforms to

$$ds^2 = 2\,du\,dr + du^2 - R^2\sin^2(r/R)(d\alpha^2 + \sin^2\alpha\,d\beta^2). \tag{1.3}$$

Bonnor and Vaidya (1970) have obtained a solution describing the field of a radiating charged particle. The metric of their solution can be expressed as

$$ds^2 = 2\,du\,dr + (1 - 2m(u)/r + 4\pi e^2(u)/r^2)\,du^2 - r^2(d\alpha^2 + \sin^2\alpha\,d\beta^2). \tag{1.4}$$

The functions $m(u)$ and $e(u)$ are respectively the mass and the charge of the particle. Here it should be noted that this metric was given without physical interpretation by Plebanski and Stachel (1967). If $m = e = 0$ the metric (1.4) becomes flat. Thus the metric (1.4) is described in the flat background. It would be interesting to obtain the metric (1.4) in the cosmological background of Einstein's static universe rather than in

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the standard Minkowskian background, and the object of the present investigation is to do just that.

We take the space surrounding the radiating charged particle to be occupied by a spherically symmetric matter distribution of nonzero density ρ and pressure p. The field equations are

$$R_{ik}-\tfrac{1}{2}g_{ik}R=-8\pi[(\rho+p)V_iV_k-pg_{ik}+\sigma\xi_i\xi_k+E_{ik}]+\Lambda g_{ik} \tag{1.5a}$$

$$E_{ik}=-g^{lm}F_{il}F_{km}+\tfrac{1}{4}g_{ik}F^{lm}F_{lm} \tag{1.5b}$$

$$F_{ik}=A_{i,k}-A_{k,i} \tag{1.5c}$$

$$F^{ik}{}_{,k}=J^i \tag{1.5d}$$

where R_{ik} is the Ricci tensor, Λ is the cosmological constant, σ is the radiation density, A_i and F_{ik} are the electromagnetic vector potential and field tensor, V_i is the flow vector of a perfect fluid, and ξ_i is the null vector satisfying

$$g^{ik}V_iV_k=1, \qquad g^{ik}\xi_i\xi_k=0 \tag{1.5e}$$

g^{ik} being the metric tensor.

In this paper we put forward a new exact solution of the above field equations.

2. The Ricci tensor

We first take the Einstein universe given by (1.3) as the background universe. Consider the following metric with signature $(-,-,-,+)$:

$$ds^2=2\,du\,dr+2L\,du^2-R^2\sin^2(r/R)(d\alpha^2+\sin^2\alpha\,d\beta^2) \tag{2.1}$$

where L is a function of r and u. Here the coordinates r, α, β and u are taken as x^1, x^2, x^3 and x^4 respectively. The explicit expressions for the nonzero components of the Ricci tensor R_{ik} for the metric (2.1) are

$$R_{11}=-2L^2/R^2-(2L_u/R)\cot(r/R)$$

$$R_{14}=-2L_{rr}-(4L_r/R\cot(r/R)$$

$$R_{22}/M^2=R_{33}/M^2\sin^2\alpha$$

$$=-2L/R^2-(1/R^2)\operatorname{cosec}^2(r/R)+(2L/R^2)\cot^2(r/R)+(2L_r/R)\cot(r/R)$$

$$R_{44}=-(2L_u/R)\cot(r/R)-2LL_{rr}-(4LL_r/R)\cot(r/R). \tag{2.2}$$

Here and in what follows a suffix denotes a partial derivative, e.g. $L_r=\partial L/\partial r$, etc.

3. The electromagnetic field

For the present problem we take the four-potential A_i to be

$$A_i=(e/R)\cot(r/R)\delta_i^4 \tag{3.1}$$

where e is an arbitrary function of u. From the Maxwell equations (1.5c) and (1.5d) we find that J^1 is the only nonzero component of the current vector J^i. It is given by

$$J^1=(e_u/R^2)\operatorname{cosec}^2(r/R). \tag{3.2}$$

It is clear from (3.2) that the current is radial and null, i.e. $g_{ik}J^iJ^k = 0$. The resulting nonzero components of E_{ik} are given by

$$\frac{E_{22}}{R^2\sin^2(r/R)} = \frac{E_{33}}{R^2\sin^2\alpha\,\sin^2(r/R)} = \frac{E_{44}}{2L} = E_{14} = \left(\frac{e^2}{2R^4}\right)\operatorname{cosec}^4\left(\frac{r}{R}\right). \tag{3.3}$$

4. The solution of the field equations

For the metric (2.1) we can take the vectors ξ_i and V_i as

$$\xi_i = (0, 0, 0, 1)$$
$$V_i = (1/2n, 0, 0, L/2n + n) \tag{4.1}$$

where n is a parameter to be determined with the help of the results (2.2), (3.3) and (4.1). The field equation (1.5a) then leads to

$$1/n^2 = 2L. \tag{4.2}$$

Here we have assumed that $2L$ is positive.

$$(1-2L)\operatorname{cosec}^2(r/R)/R^2 - 8\pi e^2\operatorname{cosec}^4(r/R)/R^4 + L_{rr} = 0. \tag{4.3}$$

The differential equation (4.3) can be easily integrated, and the solution expressed in the form

$$2L = 1 - (2m/R)\cot(r/R) + (4\pi e^2/R^4)[\cot^2(r/R) - 1] \tag{4.4}$$

where m is an arbitrary function of u.

Using (4.4) we can find explicit expressions for pressure, density and radiation density:

$$8\pi p = \Lambda - 2L/R^2 \tag{4.5}$$

$$8\pi\rho = -\Lambda + 6L/R^2 \tag{4.6}$$

$$8\pi\sigma = -\frac{2m_u}{R^2}\cot^2\left(\frac{r}{R}\right) + \frac{8\pi e e_u}{R^3}\cot\left(\frac{r}{R}\right)\left[\cot^2\left(\frac{r}{R}\right) - 1\right]. \tag{4.7}$$

From the results (4.5) and (4.6) it is clear that $24\pi p + 8\pi\rho = 2\Lambda$, i.e. $3p + \rho = \text{const}$.

The metric of our solution can be expressed in the final form as

$$\mathrm{d}s^2 = 2\,\mathrm{d}u\,\mathrm{d}r - R^2\sin^2(r/R)(\mathrm{d}\alpha^2 + \sin^2\alpha\,\mathrm{d}\beta^2)$$
$$+\left\{1 - \frac{2m}{R}\cot\left(\frac{r}{R}\right) + \frac{4\pi e^2}{R^2}\left[\cot^2\left(\frac{r}{R}\right) - 1\right]\right\}\mathrm{d}u^2. \tag{4.8}$$

When $m = e = 0$ the metric (4.8) reduces to the metric (1.3) of Einstein's universe. When R tends to infinity then (4.8) reduces to the Bonnor and Vaidya (1970) metric (1.4) describing the field of a radiating charged particle. Thus we have found a metric which in the vicinity of the source reduces to the metric describing the field of a radiating charged particle and which in the absence of the source reduces to the metric of Einstein's universe.

When $e = 0$ the metric (4.8) becomes the radiating-star metric of Vaidya (1953) in the background of Einstein's universe. Also when e and m are constants the metric

(4.8) reduces to the Nordstrom metric in the cosmological background of Einstein's universe.

Here it should be noted that our uncharged non-radiating metric is closely related to the solution obtained by Whittaker (1968). He assumed the equation of state $\rho + 3p =$ const.

It is well known that the Einstein universe given by the metric (1.3) is conformally flat. Therefore one may be tempted to believe that our results are conformally equivalent to those of Bonnor and Vaidya. But this is not true. We have verified that our metric (4.8) is not conformal to the metric (1.4) of Bonnor and Vaidya (1970).

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